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Permanence for a nonautonomous discrete single-species system with delays and feedback control☆

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ABSTRACT

A nonautonomous discrete single-species system with delays and feedback control is studied. New sufficient conditions for ensuring the permanence of the system are obtained. A very important fact is found in our results, that is, that the feedback control is harmless to the permanence of species. The corresponding results given in [F.D. Chen, Permanence of a single species discrete model with feedback control and delay, Appl. Math. Lett. 20 (2007) 729–733] are improved and extended.

1. Introduction

In this work, we study the following nonautonomous discrete single-species system with delays and feedback controls:

$$\begin{aligned} x(n+1) &= x(n)\exp\left\{r(n) - \sum_{i=1}^{m} a_i(n)x(n-\tau_i(n)) - c(n)u(n-\delta(n))\right\}, \\ u(n+1) &= (1-\eta(n))u(n) + a(n)x(n-\sigma(n)), \end{aligned} \tag{1}$$

where $x(n)$ is the density of the species at time n and $u(n)$ is the control variable at time n. For any bounded sequence $\{\zeta(n)\}$ we define $\zeta^u = \sup_{n\in Z}\{\zeta(n)\}$ and $\zeta^l = \inf_{n\in Z}\{\zeta(n)\}$, where $Z = \{1, 2, \ldots\}$.

Throughout this work, we use the following assumptions.

(A_1) $r(n)$, $a_i(n)$ $(i = 1, 2, \ldots, m)$, $c(n)$, $\eta(n)$ and $a(n)$ are nonnegative bounded sequences of real numbers defined on Z such that

$$r^l > 0,\ a_i^l > 0\ (i = 1, 2, \ldots, m),\ 0 < \eta^l \le \eta^u < 1,\ c^l \ge 0,\ a^l \ge 0.$$

(A_2) $\tau_i(n)$ $(i = 1, 2, \ldots, m)$, $\delta(n)$ and $\sigma(n)$ are nonnegative bounded integer sequences defined on Z.

Let $\tau = \max\{\tau_i(n), \delta(n), \sigma(n) : i = 1, 2, \ldots, m, n \in Z\}$. Given the biological sense, we only consider the solution of system (1) with the following initial conditions: $x(\theta) = \varphi(\theta) \ge 0$ and $u(\theta) = \psi(\theta) \ge 0$ for all $\theta = -\tau, -\tau+1, \ldots, -1$, $\varphi(0) > 0$ and $\psi(0) > 0$.

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As is well known, the discrete time population models with feedback controls have been extensively studied in many articles (see, e.g., [1–6] and references cited therein). In particular, the discrete time single-species models with feedback controls are studied in articles [1,4]. In [4], Li and Zhu proposed the following single-species discrete model with feedback control:

$$\begin{aligned} N(n+1) &= N(n)\exp\left[r(n)\left(1-\frac{N(n-m)}{k(n)}-c(n)\mu(n)\right)\right], \\ \mu(n+1) &= (1-a(n))\mu(n)+b(n)N(n-m). \end{aligned} \tag{2}$$

The assumptions made are that $a : Z \to (0, 1)$ and $c, k, r, b : Z \to R_+$ are all ω-periodic sequences and $m \geq 0$ is an integer, where Z and R_+ denote the sets of all nonnegative integers and all nonnegative real numbers. By applying the continuation theorem of coincidence degree theory, they obtained that system (2) admits at least one positive ω-periodic solution. In [1], for system (2), the permanence, as one of the most important topics in the study of population dynamics, was investigated by Chen. Chen obtained the following result.

Theorem 1. *Suppose that the following assumptions hold.*

(H_1) *$r(n)$, $k(n)$, $c(n)$, $a(n)$ and $b(n)$ are nonnegative bounded sequences of real numbers defined on Z such that*

$$r^l > 0,\ k^l > 0,\ c^l > 0,\ 0 < a^l \leq a^u < 1,\ b^l > 0;$$

(H_2) *$c^u M_2 < 1$, where $M_2 = \frac{b^u M_1}{a^l}$ and $M_1 = \frac{k^u \exp\{r^u(m+1)-1\}}{r^u}$.*

Then system (2) is permanent.

However, we see that in above theorem assumption (H_2) is very strong. In this work, our purpose is to improve the above result. We will prove that system (2) is permanent only if assumption (H_1) is satisfied. We will first consider system (1) which is more general than system (2). By further developing the analysis technique given in [7] for the nonautonomous logistic system with infinite delay, we will establish a very weak sufficient condition for the permanence of system (1) under just assumptions (A_1) and (A_2). We will find that, in our results, the feedback control is harmless to the permanence of species.

2. Main result

We first consider the following difference inequality:

$$N(n+1) \leq N(n)\exp\{\alpha(n)-\beta(n)N(n)\}, \quad n \in Z, \tag{3}$$

where $\alpha(n)$ and $\beta(n)$ are bounded sequences of real numbers defined on Z with $\alpha^l \geq 0$ and $\beta^l > 0$.

Lemma 1. *Let $N(n)$ be a nonnegative solution of inequality (3) with $N(0) > 0$; then*

$$\limsup_{n\to\infty} N(n) \leq \frac{\exp\{\alpha^u - 1\}}{\beta^l}.$$

The proof of Lemma 1 is similar to that of Theorem 2.1 given in [8] and hence we omit it here.

We further consider the following discrete linear equation:

$$N(n+1) = N(n)(1-\gamma(n)) + \omega(n), \tag{4}$$

where $\gamma(n)$ and $\omega(n)$ are bounded sequences of real numbers defined on Z with $0 < \gamma^l \leq \gamma^u < 1$ and $\omega^l \geq 0$.

Lemma 2. *Any solution $N(n)$ of system (4) with $N(0) > 0$ satisfies*

$$\frac{\omega^l}{\gamma^u} \leq \liminf_{n\to\infty} N(n) \leq \limsup_{n\to\infty} N(n) \leq \frac{\omega^u}{\gamma^l}.$$

The proof of Lemma 2 is given in [9].

Lemma 3. *For any constants $\varepsilon > 0$ and $M > 0$ there exist positive constants $\delta = \delta(\varepsilon)$ and $\hat{n} = \hat{n}(\varepsilon, M)$ such that for any $n_0 \in Z$ and $0 \leq N_0 \leq M$, when $\omega(n) < \delta$ for all $n \geq n_0$ we have*

$$N(n, n_0, N_0) < \varepsilon \quad \text{for all } n \geq n_0 + \hat{n},$$

where $N(n, n_0, N_0)$ is the solution of Eq. (4) with initial condition $N(n_0, n_0, N_0) = N_0$.

Proof. By the variation-of-constants formula for difference equations (see [9]), we have

$$\begin{aligned} N(n, n_0, N_0) &= N_0 \prod_{i=n_0}^{n-1}(1-\gamma(i)) + \sum_{i=n_0}^{n-1}\left[\prod_{s=i+1}^{n-1}(1-\gamma(s))\omega(j)\right] \\ &\le M \prod_{i=n_0}^{n-1}(1-\gamma^l) + \sum_{i=n_0}^{n-1}\left[\prod_{s=i+1}^{n-1}(1-\gamma^l)\delta\right] \\ &= \frac{(1-\gamma^l)^{n-n_0}(M\gamma^l-\delta)+\delta}{\gamma^l} \\ &\le (1-\gamma^l)^{n-n_0}M + 2\frac{\delta}{\gamma^l}. \end{aligned}$$

If we choose

$$\delta = \frac{\varepsilon\gamma^l}{4}, \qquad \hat{n} = \frac{\ln\varepsilon - \ln(2M)}{\ln(1-\gamma^l)} + 1,$$

then we have $N(n, n_0, N_0) < \varepsilon$ for all $n \ge n_0 + \hat{n}$. This completes the proof of Lemma 3. □

Theorem 2. *Assume that* (A_1) *and* (A_2) *hold. Then system* (1) *is permanent.*

Proof. Let $(x(n), u(n))$ be any positive solution of system (1); from the first equation of system (1) we have

$$x(n+1) \le x(n)\exp\{r(n)\} \quad \text{for all } n \ge 0.$$

Hence, when $n \ge \tau$, for every $i = 1, 2, \ldots, m$ we have

$$x(n) \le x(n-\tau_i(n))\exp(r^u\tau_i(n)).$$

Consequently,

$$x(n-\tau_i(n)) \ge x(n)\exp\{-r^u\tau_i(n)\} \ge x(n)\exp\{-r^u\tau\}.$$

Substituting this inequality into the first equation of system (1), we obtain

$$x(n+1) \le x(n)\exp\left\{r(n) - \exp\{-r^u\tau\}\sum_{i=1}^{m} a_i^l x(n)\right\}.$$

From Lemma 1, we further obtain that there exists a constant $\bar{x} > 0$ such that

$$\limsup_{n\to\infty} x(n) \le \bar{x}. \tag{5}$$

For any positive constant ε small enough, from (5), there exists an integer n_1 large enough that

$$x(n) < \bar{x} + \varepsilon \quad \text{for all } n \ge n_1.$$

From the second equation of system (1) we further have

$$u(n+1) \le u(n)(1-\eta(n)) + a(n)(\bar{x}+\varepsilon) \quad \text{for all } n \ge n_1 + \tau. \tag{6}$$

Since $0 < \eta^l \le \eta^u < 1$ and $a^l \ge 0$, by Lemma 2 and the comparison theorem for difference equations (see [9, Chapter 5, Theorem 2.1]), from (6) we obtain

$$\limsup_{n\to\infty} u(n) \le \frac{a^u(\bar{x}+\varepsilon)}{\eta^l}.$$

By the arbitrariness of ε in above inequality, we finally have

$$\limsup_{n\to\infty} u(n) \le \frac{a^u\bar{x}}{\eta^l} = \bar{u}. \tag{7}$$

From (A_1), we can choose a constant $\varepsilon_0 > 0$ small enough that

$$r^l - \varepsilon_0 c^u \ge \varepsilon_0. \tag{8}$$

Consider the following auxiliary equation:

$$v(n+1) = v(n)(1-\eta(n)) + a(n)\alpha_0, \tag{9}$$

where α_0 is a parameter. From Lemma 3, for $\varepsilon_0 > 0$ given above and positive constant $M = \max\{\bar{x}, \bar{u}\} + 1$, there exist constants $\delta_0 = \delta_0(\varepsilon_0)$ and $\hat{n}_0 = \hat{n}_0(\varepsilon_0, M)$ such that for any $n_0 \in Z$ and $0 \le v_0 \le M$, when $\alpha_0 a(n) < \delta_0$, we have

$$v(n, n_0, v_0) < \varepsilon_0 \quad \text{for all } n \ge n_0 + \hat{n}_0,$$

where $v(n, n_0, v_0)$ is the solution of Eq. (9) with initial condition $v(n_0, n_0, v_0) = v_0$.

It follows from (8) that there exists a positive constant $\alpha_0 \le \min\{\varepsilon_0, \frac{\delta_0}{a^u+1}\}$ such that $\frac{a^u\alpha_0}{\eta^l} < \varepsilon_0$ and

$$r^l - \sum_{i=1}^{m} a_i^u\alpha_0 - \varepsilon_0 c^u \ge \alpha_0. \tag{10}$$

We first prove that

$$\limsup_{n\to\infty} x(n) \ge \alpha_0 \tag{11}$$

for any positive solution $(x(n), u(n))$ of system (1). In fact, if (11) is not true, then there exists a positive solution $(x(n), u(n))$ of system (1) and an integer $\hat{n}_1 > 0$ such that $x(n) < \alpha_0$ for all $n \ge \hat{n}_1$. Then, we have

$$u(n+1) \le u(n)(1 - \eta(n)) + a(n)\alpha_0 \quad \text{for all } n \ge \hat{n}_2 + \tau.$$

By an argument similar to that in the proof of inequality (7), we obtain

$$\limsup_{n\to\infty} u(n) \le \frac{a^u\alpha_0}{\eta^l}.$$

Hence, there exists an integer $\hat{n}_2 \ge \hat{n}_1$ such that

$$u(n) < \varepsilon_0 \quad \text{for all } n \ge \hat{n}_2.$$

Thus, for any $n \ge \hat{n}_2 + \tau$, from the first equation of system (1) we obtain

$$\begin{aligned} x(n+1) &\ge x(n)\exp\left\{r^l - \sum_{i=1}^{m} a_i^u\alpha_0 - c^u\varepsilon_0\right\} \\ &\ge x(n)\exp\{\alpha_0\}. \end{aligned}$$

From this, we further obtain $x(n) \to \infty$ as $n \to \infty$, which is contradictory to the boundedness of solutions of system (1). Therefore, (11) holds.

Next, we prove that there exists a positive constant $\underline{x}$ such that

$$\liminf_{n\to\infty} x(n) \ge \underline{x} \tag{12}$$

for any positive solution $(x(n), u(n))$ of system (1). In fact, if (12) is not true, then there is a sequence of initial values $z^{(k)} = (\phi^{(k)}, \psi^{(k)})$ such that

$$\liminf_{n\to\infty} x(n, z^{(k)}) < \frac{\alpha_0}{k^2} \quad \text{for all } k = 1, 2, \ldots,$$

where $(x(n, z^{(k)}), u(n, z^{(k)}))$ is the solution of system (1) with initial condition $x(n) = \phi^{(k)}(n)$ and $u(n) = \psi^{(k)}(n)$ for all $n \in [-\tau, 0]$. From (11), we obtain that, for every $k \in Z$, there exist two time sequences $\{s_q^{(k)}\}$ and $\{t_q^{(k)}\}$ such that $0 < s_1^{(k)} < t_1^{(k)} < s_2^{(k)} < t_2^{(k)} < \cdots < s_q^{(k)} < t_q^{(k)} < \cdots$ and $s_q^{(k)} \to \infty$ as $q \to \infty$ such that

$$x(s_q^{(k)}, z^{(k)}) > \frac{\alpha_0}{k}, \qquad x(t_q^{(k)}, z^{(k)}) < \frac{\alpha_0}{k^2} \tag{13}$$

and

$$\frac{\alpha_0}{k^2} \le x(n, z^{(k)}) \le \frac{\alpha_0}{k} \quad \text{for all } n \in [s_q^{(k)} + 1,\ t_q^{(k)} - 1]. \tag{14}$$

From (5) and (7), for every $k \in Z$, there exists an integer $\hat{n}_2^{(k)}$ such that

$$x(n, z^{(k)}) \le M, \qquad u(n, z^{(k)}) \le M \quad \text{for all } n \ge \hat{n}_2^{(k)}.$$

Further, from $\lim_{q\to\infty} s_q^{(k)} = \infty$, there exists an integer $N_1^{(k)} > 0$ such that $s_q^{(k)} > \hat{n}_2^{(k)} + \tau$ for all $q \ge N_1^{(k)}$. For any $n \in [s_q^{(k)}, t_q^{(k)}]$ and $q \ge N_1^{(k)}$, from the first equation of system (1) we have

$$x(n+1, z^{(k)}) \ge x(n, z^{(k)})\exp\{-\theta\},$$

where $\theta = r^u + \sum_{i=1}^{m} a^u M + c^u M$. Hence,

$$x(t_q^{(k)}, z^{(k)}) \geq x(s_q^{(k)}, z^{(k)}) \exp\{-\theta(t_q^{(k)} - s_q^{(k)})\},$$

which implies from (13)

$$t_q^{(k)} - s_q^{(k)} > \frac{\ln k}{\theta} \quad \text{for all } q \geq N_1^{(k)}.$$

Choose $K_0 > 0$ such that

$$t_q^{(k)} - s_q^{(k)} > \hat{n}_0 + 2\tau \quad \text{for all } k \geq K_0,\ q \geq N_1^{(k)}.$$

For any $k \geq K_0, q \geq N_1^{(k)}$ and $n \in [s_q^{(k)} + \tau, t_q^{(k)}]$, from the second equation of system (1) and (14) we have

$$u(n+1, z^{(k)}) \leq (1 - \eta(n))u(n, z^{(k)}) + \alpha_0 a(n). \tag{15}$$

Assume that $v(n)$ is the solution of Eq. (9) with initial condition $v(s_q^{(k)} + \tau) = u(s_q^{(k)} + \tau)$; then from the comparison theorem for difference equations and inequality (15), we obtain

$$u(n, z^{(k)}) \leq v(n) \quad \text{for all } n \in [s_q^{(k)} + \tau, t_q^{(k)}]. \tag{16}$$

In Lemma 3, choosing $n_0 = s_q^{(k)} + \tau$ and $v_0 = u(s_q^{(k)} + \tau)$, since $0 \leq v_0 \leq M$ and $\alpha_0 a(n) < \delta_0$ for all $n \in [s_q^{(k)} + \tau, t_q^{(k)}]$, we have

$$v(n) = v(n, s_q^{(k)} + \tau, u(s_q^{(k)} + \tau)) < \varepsilon_0 \quad \text{for all } n \in [s_q^{(k)} + \hat{n}_0 + \tau, t_q^{(k)}].$$

Hence, from (16) we further have

$$u(n, z^{(k)}) < \varepsilon_0 \quad \text{for all } n \in [s_q^{(k)} + \hat{n}_0 + \tau, t_q^{(k)}],\ k \geq K_0, q \geq N_1^{(k)}.$$

For any $n \in [s_q^{(k)} + \hat{n}_0 + 2\tau,\ t_q^{(k)}]$ and $k \geq K_0, q \geq N_1^{(k)}$, from the first equation of system (1), (10) and (17) we further have

$$\begin{aligned} x(n+1, z^{(k)}) &\geq x(n, z^{(k)}) \exp\left\{ r^l - \sum_{i=1}^{m} a_i^u \alpha_0 - c^u \varepsilon_0 \right\} \\ &\geq x(n, z^{(k)}) \exp(\alpha_0). \end{aligned}$$

Hence,

$$x(t_q^{(k)}, z^{(k)}) \geq x(t_q^{(k)} - 1, z^{(k)}) \exp(\alpha_0),$$

In view of (13) and (14), we finally have

$$\frac{\alpha_0}{k^2} > \exp\{\alpha_0\} \frac{\alpha_0}{k^2},$$

which is contradictory. Therefore, (12) holds. This shows that system (1) is permanent. This completes the proof of Theorem 1. □

Applying Theorem 2 to system (2), we have the following corollary.

Corollary 1. *Assume that* (H_1) *holds; then system (2) is permanent.*

Obviously, Corollary 1 is a very good improvement of Theorem 1. We see that assumption (H_2) is superfluous to ensuring the permanence of system (2).

Remark 1. From Theorem 2 we directly see that for system (1) the feedback control is harmless to the permanence of species.

Lastly, an example is provided to illustrate our results. We consider the following nonautonomous discrete single-species system with delays and feedback control:

$$\begin{aligned} N(n+1) &= N(n) \exp\left\{ (2 + \sin(n)) \left(1 - \frac{N(n-1)}{3 + \cos(n)} - (2 + \sin(n))\mu(n) \right) \right\}, \\ \mu(n+1) &= \left(\frac{1}{2} - \frac{1}{4^n} \right) \mu(n) + (4 + \sin(n))N(n-1). \end{aligned} \tag{17}$$

Obviously, for system (17), assumptions (A_1) and (A_2) are satisfied. Therefore, from Theorem 2 we obtain that system (17) is permanent. But, by calculating we can obtain

$$M_1 = \frac{k^u \exp\{r^u(m+1)-1\}}{r^u} \geq \frac{2e}{3}, \qquad M_2 = \frac{b^u M_1}{a^l} \geq 2e, \qquad c^u M_2 \geq 2e.$$

This shows that system (17) does not satisfy assumption (H_2) in Theorem 1.

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